

# RATIONALITY OF MODULI SPACE OVER REDUCIBLE CURVE

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**ABSTRACT.** Let  $M(2, \underline{\mathbf{w}}, \chi)$  be the moduli space of rank 2 torsion-free sheaves over a reducible nodal curve with each component having utmost two nodal singularities. We show that in each component of  $M(2, \underline{\mathbf{w}}, \chi)$ , the closure of rank 2 vector bundles with fixed determinant is rational.

## 1. INTRODUCTION

Let  $C$  be a connected projective curve over an algebraically closed field  $k$  of characteristic 0 having  $N$  smooth components  $C_i$  of genus  $g_i \geq 2$  and  $N - 1$  nodes  $P_i$  such that  $C_i \cap C_{i+1} = P_i$ , for  $i = 1, 2, \dots, N - 1$ . We call such a curve as a *chain-like curve*. Let  $\underline{\mathbf{w}} := (w_1, w_2, \dots, w_N)$  be an  $N$ -tuple of positive rational numbers such that  $\sum_{j=1}^N \omega_j = 1$ , we call this a *polarisation* on  $C$ . Let  $\chi$  be an odd integer and  $M(2, \underline{\mathbf{w}}, \chi)$  be the moduli space of rank 2,  $\underline{\mathbf{w}}$ -semi-stable torsion free sheaves with fixed Euler characteristic  $\chi$  [13]. It is known that for a generic choice of  $\underline{\mathbf{w}}$ ,  $M(2, \underline{\mathbf{w}}, \chi)$  has  $2^{N-1}$  irreducible components  $M_l(2, \underline{\mathbf{w}}, \chi)$ ,  $l = 1$  to  $2^{N-1}$  [11]. Each component is determined by choosing an  $N$ -tuple of integers  $(\chi_1, \chi_2, \dots, \chi_N)$  satisfying inequalities (2.4) and (2.2) (for  $n=2$ ) such that for a generic vector bundle  $E$  in the component,  $\chi(E|_{C_i}) = \chi_i$ .

Let  $\xi$  be a line bundle on  $C$  given by  $(L_1, L_2, \dots, L_N)$ , where  $L_i$ 's are invertible sheaves on  $C_i$ 's respectively. Let  $\overline{M}_l(2, \underline{\mathbf{w}}, \chi, \xi)$  denote the closure of collection of vector bundles with determinant  $\xi$  in  $M_l(2, \underline{\mathbf{w}}, \chi)$ . In this article we want to prove that this subvariety  $\overline{M}_l(2, \underline{\mathbf{w}}, \chi, \xi)$  is rational for each  $l$ . When  $N = 2$  this result has appeared in [1] and these subvarieties are interpreted as fixed determinant moduli space of torsion free sheaves [2], [6]. When  $N > 2$  such an analogue of fixed determinant moduli space of torsion free sheaves is not known. We expect that if we have a similar notion of fixed determinant moduli space, then our result will tell that it will be rational.

Over a smooth projective curve of genus  $g \geq 2$ , the rationality of the moduli space was first proved by Tjurin [14, Theorem 14] in the rank 2 and odd degree case. When rank and degree are coprime this result was generalized by Newstead [8], [9], King and Schofield [4] in higher order of generalities. It is still not known if the moduli space is rational or not in the non-coprime case, even for rank 2 and degree 0. In the non-smooth case, when the curve is irreducible and has any number of nodal singularities and genus  $\geq 2$ , rationality in the coprime case was proved by Bhosle and Biswas [3, Theorem 3.7]. Over a reducible nodal curve with two components (i.e.  $N = 2$ ) the moduli space of torsion free sheaves with fixed determinant has two components [2]. The proof of rationality of each of these components given in [1] uses Nagaraj-Seshadri's description of the moduli space in terms of triples [6]. For higher values of  $N$ , such a description is not known. Hence the proof given in [1] can

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not be generalized for arbitrary  $N$ . The proof in this article is based on Newstead's idea [8] and Teixidor i Bigas's description of the moduli space [11] but involves several technical challenges. In fact Teixidor i Bigas's description of the moduli space holds for more general curve known as *tree-like curve* but the combinatorics involved will be more complicated. It will be interesting as well as challenging to investigate rationality question in this case.

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## 2. DESCRIPTION OF THE MODULI SPACE

Let  $C$  be a chain-like curve with  $N$  irreducible components  $C_i$  of genus  $g_i \geq 2$  such that  $C_i \cap C_{i+1} = P_i$ , for  $i = 1, 2, \dots, N-1$ . Suppose  $E$  is a vector bundle on  $C$  of rank  $n$  and Euler characteristic  $\chi$  and  $E_i$  is  $E|_{C_i}$ . Then one has the following exact sequence -

$$0 \rightarrow E \rightarrow \bigoplus_{j=1}^N \alpha_{j*}(E_j) \rightarrow T_E \rightarrow 0, \quad (2.1)$$

where  $\alpha_j : C_j \rightarrow C$  is a closed immersion for each  $j$  and  $T_E$  is a torsion sheaf supported only at the nodal points. Let  $\chi_j := \chi(E_j)$ . Then from the exact sequence (2.1), one can show that

$$\chi = \sum_{j=1}^N \chi_j - n(N-1). \quad (2.2)$$

Now let  $\underline{w} := (w_1, w_2, \dots, w_N)$  be a polarization on  $C$ , i.e.,  $w_j$  is a positive rational number for each  $j$ , and  $\sum_{j=1}^N w_j = 1$ . For any torsion-free sheaf  $E$  on  $C$ , we define

$$\mu(E) = \frac{\chi(E)}{\sum_{j=1}^N w_j \operatorname{rk}(E_j)}, \quad (2.3)$$

where  $E_j = \frac{E|_{C_j}}{\text{torsion}}$ .

**Definition 2.1.** We say that a torsion-free sheaf  $E$  on  $C$  is *stable* (resp. *semi-stable*) if for every proper sub-sheaf  $G$  of  $E$ , we have  $\mu(G) < \mu(E)$  (resp.  $\leq$ ).

It is a theorem of Seshadri that over any reducible curve, the moduli space  $M(n, \underline{w}, \chi)$  of semi-stable torsion-free sheaves of rank  $n$  and euler characteristic  $\chi$  exists and is compact (see [13, Chap VII]).

In the case of a chain-like curve  $C$  with  $N$  components, Teixidor i Bigas has proven in [11, Theorem-1, Step-1] that  $M(n, \underline{w}, \chi)$  has  $n^{N-1}$  components for any generic polarization.

<sup>1</sup> In fact she has shown that if  $E$  is a stable vector bundle on  $C$  with Euler characteristic  $\chi$  and  $E_i$  has Euler characteristic  $\chi_i$  for each  $i$ , then  $\chi_i$ 's are going to satisfy the inequalities :  
<sup>2</sup>

$$\left(\sum_{j=1}^i w_j\right)\chi - \sum_{j=1}^{i-1} \chi_j + n(i-1) < \chi_i < \left(\sum_{j=1}^i w_j\right)\chi - \sum_{j=1}^{i-1} \chi_j + ni, \quad (2.4)$$

<sup>1</sup>In fact she proves this result for tree-like curves.

<sup>2</sup>The inequalities (2.4) follow from [11, Theorem-1, Step-1, (1)].

for  $i = 1, 2, \dots, N-1$ , provided  $(\sum_{j=1}^i w_j)\chi$  is not an integer for each  $i \in \{1, 2, \dots, N-1\}$ . She also proves in [11, Theorem-1, Step-2] that, for any choice of a semi-stable vector bundle  $E_i$  on each component  $C_i$  with Euler characteristic  $\chi_i$  as in the inequality (2.4), and gluing by any isomorphism at the nodes, one can obtain a semi-stable vector bundle  $E$  on  $C$  and further if one of the  $E_i$  is stable, so is  $E$ . Since there are  $n^{N-1}$  choices for such tuples  $(\chi_1, \chi_2, \dots, \chi_N)$ , one can conclude that  $M(n, \mathbf{w}, \chi)$  has  $n^{N-1}$  components, each component corresponding to a particular type of  $(\chi_1, \chi_2, \dots, \chi_N)$ .

In what follows, we assume that  $\chi$  is odd and  $n = 2$ . We also choose the polarization  $\mathbf{w}$  in such a way that stability coincides with semi-stability.

### 3. CONSTRUCTION OF A STABLE FAMILY

Let  $\chi = 1$  and  $\chi_1, \chi_2, \dots, \chi_{N-1}$  be integers satisfying the inequalities

$$\left(\sum_{j=1}^i w_j\right) - \sum_{j=1}^{i-1} \chi_j + 2(i-1) < \chi_i < \left(\sum_{j=1}^i w_j\right) - \sum_{j=1}^{i-1} \chi_j + 2i. \quad (3.1)$$

Let  $\chi_N$  be an integer which fits into the following equation-

$$\chi = 1 = \sum_{j=1}^N \chi_j - 2(N-1). \quad (3.2)$$

So there are  $2^{N-1}$  choices for the tuple  $(\chi_1, \chi_2, \dots, \chi_N)$  satisfying 3.1 and 3.2. By the equation (3.2), for all these choices one has

$$\sum_{j=1}^N \chi_j = 2N-1. \quad (3.3)$$

Let  $L_j$  be an invertible sheaf on  $C_j$  for each  $j \in \{1, 2, \dots, N\}$ .

**Definition 3.1.** We say that the tuple  $(L_1, L_2, \dots, L_N)$  is of type  $(\chi_1, \chi_2, \dots, \chi_N)$  if  $\deg(L_j) = \chi_j - 2(1 - g_j)$  for each  $j$ .

Throughout this section we fix an invertible sheaf  $L_j$  on  $C_j$  for each  $j$ , such that  $L_j$ 's are globally generated and the tuple  $(L_1, L_2, \dots, L_N)$  is of type  $(\chi_1, \chi_2, \dots, \chi_N)$  where  $\chi_1, \chi_2, \dots, \chi_{N-1}$  are as in 3.1 and  $\chi_N$  is as in the equation (3.2).

Let

$$T_j = \{t \in H^0(C_j, L_j) \mid t(P_{j-1}) \neq 0 \text{ and } t(P_j) \neq 0\},$$

for  $j = 2, \dots, N-1$ . Similarly let

$$T_1 = \{t \in H^0(C_1, L_1) \mid t(P_1) \neq 0\},$$

and

$$T_N = \{t \in H^0(C_N, L_N) \mid t(P_{N-1}) \neq 0\}.$$

Clearly for each  $j \in \{1, \dots, N\}$ ,  $T_j$  is a non-empty Zariski-open subset of the affine space  $H^0(C_j, L_j)$ . So there are sections in  $H^0(C_j, L_j)$  which do not vanish on any node of  $C_j$ . Let  $s_j \in H^0(C_j, L_j)$  be one such section for each  $j$ .

Let

$$\lambda_j : L_j(P_j) \rightarrow L_{j+1}(P_j)$$

be the linear map of one dimensional vector spaces which sends  $s_j(P_j)$  to  $s_{j+1}(P_j)$ , where  $j = 1, 2, \dots, N-1$ . We now define an invertible sheaf  $L$  on  $X$  as follows:

$$L = \{(t_1, t_2, \dots, t_N) \in \bigoplus_{j=1}^N \alpha_{j*}(L_j) \mid t_j(P_j) \xrightarrow{\lambda_j} t_{j+1}(P_j)\}.$$

Clearly  $(s_1, s_2, \dots, s_N) \in H^0(C, L)$  where  $s_j$ 's are as defined above. From now on, we call this section as the "distinguished section". By definition of  $L$  we have the following short exact sequence:

$$0 \rightarrow L \rightarrow \bigoplus_{j=1}^N \alpha_{j*}(L_j) \rightarrow T \rightarrow 0, \quad (3.4)$$

where  $T$  is the torsion sheaf which is supported at the nodal points. Since there are  $(N-1)$  nodes,  $H^0(C, T)$  is a vector space of dimension  $N-1$ .

**Remark 3.2.** Let  $L = (L_1, \dots, L_N)$  be as above. The fact that  $\chi_j$ 's are chosen as in 3.1 and (3.2) will imply that  $\deg(L_j) \geq 2g_j - 1$  for each  $j$ . If for some  $j$   $\deg(L_j) = 2g_j - 1$ , then such an  $L_j$  need not be globally generated in general. But one can always choose an invertible sheaf  $L_j$  of degree  $2g_j - 1$  such that  $L_j$  is globally generated (see [1, Remark 3.2(a)]).

**Lemma 3.3.** Let  $L$  be as above. Then

- (i) The functor  $H^0(C, -)$  applied to (3.4) is exact.
- (ii)  $\dim(H^0(C, L)) = g$  and  $\dim(H^1(C, L)) = 0$ .
- (iii)  $\dim(H^0(C, L^*)) = 0$  and  $\dim(H^1(C, L^*)) = 3g - 2$ , where  $L^*$  is the dual of  $L$ .

*Proof.* Applying the functor  $H^0(C, -)$  to (3.4), we get the exact sequence

$$0 \rightarrow H^0(C, L) \rightarrow \bigoplus_{j=1}^N H^0(C_j, L_j) \xrightarrow{\beta} H^0(C, T). \quad (3.5)$$

We claim  $\beta$  is surjective. Consider the set  $D = \{(s_1, 0, \dots, 0), (0, s_2, 0, \dots, 0), \dots, (0, 0, \dots, 0, s_{N-1}, 0)\}$ , where  $s_j$ 's are the components of the "distinguished section". Clearly this is a linearly independent set in  $\bigoplus_{j=1}^N H^0(C_j, L_j)$ . Since the "distinguished section" goes to zero under  $\beta$ , it is clear that image of each element of  $D$  under  $\beta$  is non-zero and in fact  $\beta(D)$  is linearly independent in  $H^0(C, T)$ . Hence  $\beta$  is surjective. This proves (i).

Now by the choice of  $L_j$ 's it is clear that  $\dim(\bigoplus_{j=1}^N H^0(C_j, L_j)) = g + (N-1)$  and  $\dim(\bigoplus_{j=1}^N H^1(C_j, L_j)) = 0$ . So by (i) and (3.5),  $\dim(H^0(C, L)) = g$  and by taking the long exact sequence associated to (3.4), we can conclude that  $\dim(H^1(C, L)) = 0$ . This proves (ii).

To prove (iii), again by the choice of  $L_j$ 's, it is clear that  $\deg(L_j^*) < 0$  for each  $j$ . So  $H^0(C_j, L_j^*) = 0$  for each  $j$ . Since  $L^* \hookrightarrow \bigoplus_{j=1}^N \alpha_{j*}(L_j^*)$ , it is clear that  $H^0(C, L^*) = 0$ . So

$$\begin{aligned}
\dim (H^1(C, L^*)) &= -\chi(L^*) \\
&= -(\chi(\bigoplus_{j=1}^N \alpha_{j*}(L_j^*)) - (N-1)) \\
&= -((\sum_{j=1}^N (\deg(L_j^*) + (1 - g_j))) - (N-1)) \\
&= 3g - 2.
\end{aligned}$$

□

Now let  $\gamma$  be a proper subset of  $\{1, 2, \dots, N\}$ . Let

$$V_\gamma = \{(t_1, t_2, \dots, t_N) \in H^0(C, L) \mid t_i \neq 0, \text{ if } i \in \gamma \text{ and } t_i = 0 \text{ otherwise}\}. \quad (3.6)$$

Clearly  $\overline{V}_\gamma$ , the closure of  $V_\gamma$  in  $H^0(C, L)$ , is a linear subspace of  $H^0(C, L)$  and hence is closed and irreducible in Zariski topology.

**Lemma 3.4.** *Let  $\gamma = \{i_1, i_2, \dots, i_t\}$  be a proper subset of  $\{1, 2, \dots, N\}$  such that  $i_1, i_2, \dots, i_t$  are consecutive integers. If  $i_1 = 1$  or  $i_t = N$ ,*

$$\dim(\overline{V}_\gamma) \leq \sum_{j=1}^t h^0(C_{i_j}, L_{i_j}) - t.$$

Otherwise

$$\dim(\overline{V}_\gamma) \leq \sum_{j=1}^t h^0(C_{i_j}, L_{i_j}) - (t+1).$$

*Proof.* If  $i_1 = 1$  or  $i_t = N$ , the union  $C_{i_1} \cup \dots \cup C_{i_t}$  has  $t-1$  internal nodes and one external node. If  $i_1 \neq 1$  and  $i_t \neq N$ , the union  $C_{i_1} \cup \dots \cup C_{i_t}$  has  $t-1$  internal nodes and two external nodes. So by the definition of  $\overline{V}_\gamma$ , the Lemma follows. □

Let

$$V = \bigcup_{\gamma} \overline{V}_\gamma. \quad (3.7)$$

Since the "distinguished section" belongs to  $H^0(C, L) \setminus V$ , we can conclude that  $V$  is a proper closed subset of the affine space  $H^0(C, L)$ . Let

$$R = H^0(C, L) \setminus V. \quad (3.8)$$

Clearly every element of  $R$  defines an injective map  $\mathcal{O}_C \hookrightarrow L$  and conversely if any non-zero section  $(t_1, t_2, \dots, t_N)$  of  $L$  defines an injective map  $\mathcal{O}_C \hookrightarrow L$ , then such a section should be in  $R$ , for otherwise, it belongs to  $V$  which means  $t_i = 0$  for some  $i$ , and so such a section  $(t_1, t_2, \dots, t_N)$  cannot define an injective map  $\mathcal{O}_C \hookrightarrow L$ . So we have

$$R = \{\psi \in H^0(C, L) \mid \psi : \mathcal{O}_C \hookrightarrow L \text{ is injective}\}. \quad (3.9)$$

**Lemma 3.5.** *Let  $\gamma = \{i_1, i_2, \dots, i_t\}$  be a proper subset of  $\{1, 2, \dots, N\}$  such that  $i_1 < i_2 < \dots < i_t$ . Then*

$$\dim(\overline{V}_\gamma) \leq \sum_{j=1}^t g_{i_j}.$$

As a consequence,

$$\dim (V_\gamma) \leq \sum_{j=1}^t g_{i_j}.$$

*Proof.* By Reimann-Roch theorem and the choice of the invertible sheaves  $L_j$ , we know that

$$\begin{aligned} \sum_{j=1}^t h^0(C_{i_j}, L_{i_j}) &= \sum_{j=1}^t [\deg (L_{i_j}) + (1 - g_{i_j})] \\ &= \sum_{j=1}^t [\chi_{i_j} - 2(1 - g_{i_j}) + (1 - g_{i_j})] \\ &= \sum_{j=1}^t \chi_{i_j} - t + \sum_{j=1}^t g_{i_j}. \end{aligned} \quad (3.10)$$

With this in mind, we prove the Lemma by considering two different cases for  $\gamma$ .

**Case A:** We first assume that  $\gamma$  consists of consecutive integers. We now consider three sub-cases -

**Case i:** Suppose  $i_1 = 1$ . This implies  $i_t = t$ . Now since the  $\chi_j$ 's satisfy (3.1),  $\sum_{j=1}^t \chi_j$  is either equal to  $2t$  or equal to  $2t - 1$ . In any case,

$$\sum_{j=1}^t \chi_j \leq 2t. \quad (3.11)$$

Therefore by the equation (3.10) and the inequality (3.11),

$$\begin{aligned} \sum_{j=1}^t h^0(C_j, L_j) &\leq 2t - t + \sum_{j=1}^t g_j \\ &= \sum_{j=1}^t g_j + t. \end{aligned} \quad (3.12)$$

Combining the inequality 3.12 and the Lemma 3.4, we have

$$\dim (\overline{V}_\gamma) \leq \sum_{j=1}^t g_j,$$

which proves the Lemma for this case.

**Case ii:** Suppose  $i_t = N$ . This implies  $i_1 - 1 = N - t$ . Now from the equation

$$\sum_{i=1}^N \chi_i = \sum_{i=1}^{i_1-1} \chi_i + \sum_{i=i_1}^N \chi_i,$$

we have

$$\sum_{i=i_1}^N \chi_i = \sum_{i=1}^N \chi_i - \sum_{i=1}^{i_1-1} \chi_i. \quad (3.13)$$

The choice of  $\chi_i$ 's will imply that the sum  $\sum_{i=1}^{i_1-1} \chi_i$  has to be at least  $2i_1 - 3$ . Using this fact and the equation (3.3) in the equation (3.13), we get

$$\begin{aligned} \sum_{i=i_1}^N \chi_i &\leq (2N-1) - (2i_1-3) \\ &= (2N-1) - 2(N-t+1) + 3 \\ &= 2t. \end{aligned} \tag{3.14}$$

Combining this with the equation (3.10), we get

$$\begin{aligned} \sum_{j=1}^t h^0(C_{i_j}, L_{i_j}) &\leq 2t - t + \sum_{j=1}^t g_{i_j} \\ &= \sum_{j=1}^t g_{i_j} + t. \end{aligned} \tag{3.15}$$

From the inequality 3.15 and the Lemma 3.4, we have

$$\dim(\overline{V}_\gamma) \leq \sum_{j=1}^t g_{i_j}.$$

**Case iii:** Suppose  $i_1 \neq 1$  and  $i_t \neq N$ . Then since

$$\sum_{i=1}^{i_t} \chi_i = \sum_{i=1}^{i_1-1} \chi_i + \sum_{i=i_1}^{i_t} \chi_i,$$

we have

$$\begin{aligned} \sum_{i=i_1}^{i_t} \chi_i &= \sum_{i=1}^{i_t} \chi_i - \sum_{i=1}^{i_1-1} \chi_i \\ &\leq 2i_t - (2i_1 - 3) \\ &= 2(i_1 + (t-1)) - (2i_1 - 3) \\ &= 2t + 1. \end{aligned} \tag{3.16}$$

So combining with the equation (3.10), we get

$$\sum_{j=1}^t h^0(C_{i_j}, L_{i_j}) \leq \sum_{j=1}^t g_{i_j} + (t+1). \tag{3.17}$$

So by the inequality 3.17 and the Lemma 3.4, we can conclude that

$$\dim(\overline{V}_\gamma) \leq \sum_{j=1}^t g_{i_j}.$$

This proves the Lemma for **Case A**.

**Case B:** Now suppose  $\gamma$  is such that  $i_1, i_2, \dots, i_t$  are not consecutive. Let  $\mathcal{C}_1, \dots, \mathcal{C}_r$  be the connected components of  $C_{i_1} \cup \dots \cup C_{i_t}$ . Clearly each  $\mathcal{C}_i$  consists of either single irreducible

component or some consecutive irreducible components of  $C$ . Let the corresponding subset of indices be  $\gamma_i$ . Then  $\gamma_i$  is either singleton or has consecutive integers. Therefore

$$\begin{aligned} \dim(\overline{V_\gamma}) &= \sum_{\gamma_i \in \mathcal{C}_i} \dim(\overline{V_{\gamma_i}}) \\ &\leq \sum_{i_j \in \gamma} g_{i_j}. \end{aligned} \quad (3.18)$$

The last inequality comes by **Case A**. This proves the Lemma.  $\square$

**Remark 3.6.** Let  $q = 3g - 2$ . Then by fixing a basis of  $H^1(C, L^*)$ , we can identify it with  $k^q$ . We have the natural  $k^*$ -action on  $k^q$  and

$$W = \{(a_1, a_2, \dots, a_q) \in k^q \mid a_1 \neq 0\}$$

is clearly an invariant Zariski-open subset of  $k^q$  under this  $k^*$ -action.

Let  $A := \{(a_1, a_2, \dots, a_q) \in W \mid a_1 = 1\}$  (Clearly  $A$  is Zariski closed and every orbit of  $k^*$ -action on  $W$  meets  $A$  in exactly one point).

**Lemma 3.7.** (cf. [5], [7], [8]) Let  $L$  be as above. Then there exists a vector space  $V'$  and a universal extension

$$0 \rightarrow \mathcal{O}_{C \times V'} \rightarrow \tilde{\mathcal{E}} \rightarrow \pi^*(L) \rightarrow 0 \quad (3.19)$$

of bundles over  $C \times V'$  (where  $\pi : C \times V' \rightarrow C$  is the projection map), such that there is a natural isomorphism

$$\alpha : V' \rightarrow H^1(C, L^*)$$

where for each  $v \in V'$ ,  $\alpha(v)$  is the element corresponding to the restriction of the extension (3.19) to  $\{v\} \times C$ .

**Remark 3.8.** Suppose  $\tilde{\mathcal{E}}$  is as in Lemma 3.7 and  $v \in H^1(C, L^*)$  is such that  $\dim(H^0(C, \tilde{\mathcal{E}}_v)) = 1$ . Then one can easily see that for any  $w \in H^1(C, L^*)$ ,  $\tilde{\mathcal{E}}_v \cong \tilde{\mathcal{E}}_w$  if and only if  $v$  and  $w$  are in the same orbit under the natural action of  $k^*$  on  $H^1(C, L^*)$ .

**Lemma 3.9.** Let  $L$  be as above. Then there exists an extension

$$0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow L \rightarrow 0, \quad (3.20)$$

for which  $\dim(H^0(C, E)) = 1$  and this extension can be chosen to correspond to a point of  $A$ , where  $A$  is as in Remark 3.6 above.

*Proof.* Suppose  $a \in H^1(C, L^*)$  and (3.20) is the corresponding extension. For any section  $0 \neq \delta \in H^0(C, L)$ , we have a non-trivial morphism

$$\delta : \mathcal{O}_C \longrightarrow L. \quad (3.21)$$

Tensoring (3.21) by the dualizing sheaf  $\omega_C$  and applying the global section functor, we get the map

$$H^0(C, \omega_C) \longrightarrow H^0(C, L \otimes \omega_C).$$

Taking dual and using the duality theorem, we get the map

$$H^1(C, L^*) \xrightarrow{\tilde{\delta}} H^1(C, \mathcal{O}_C).$$

This implies

$$\dim(\ker(\tilde{\delta})) \geq \dim(H^1(C, L^*)) - g > 0.$$



Applying the sheaf functors  $\mathcal{H}om(L, -)$  and  $\mathcal{H}om(\mathcal{O}_C, -)$  to (3.20) and taking the long exact sequence, we get the following commutative diagram -

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}om(L, \mathcal{O}_C) & \longrightarrow & \mathcal{H}om(L, E) & \longrightarrow & \mathcal{H}om(L, L) & \longrightarrow & H^1(C, L^*) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \tilde{\delta} & & \\ 0 & \longrightarrow & H^0(C, \mathcal{O}_C) & \longrightarrow & H^0(C, E) & \longrightarrow & H^0(C, L) & \longrightarrow & H^1(C, \mathcal{O}_C) & \longrightarrow & \dots \end{array}$$

From this diagram, it is clear that  $\delta$  lifts to a section on  $E$  if and only if  $\tilde{\delta}(a) = 0$ . This fact is proved in [7, Lemma 3.1], in greater generality. Also

$$\dim(H^0(C, E)) = \dim(H^0(C, \mathcal{O}_C)) + \dim(\ker(H^0(C, L) \rightarrow H^1(C, \mathcal{O}_C))).$$

(One can prove that  $\dim(H^0(C, \mathcal{O}_C)) = 1$  and  $\dim(H^1(C, \mathcal{O}_C)) = g$  by using the arguments similar to those in Lemma 3.3). So

$$\begin{aligned} \dim(H^0(C, E)) > 1 &\Leftrightarrow \dim(\ker(H^0(C, L) \rightarrow H^1(C, \mathcal{O}_C))) \geq 1 \\ &\Leftrightarrow a \in p_1(Y), \end{aligned}$$

where  $Y$  is a subset of  $H^1(C, L^*) \times H^0(C, L)$  defined by  $(a, \delta) \in Y \Leftrightarrow 0 \neq \delta$  and  $\tilde{\delta}(a) = 0$ , and  $p_1$  is the first projection.

We want an extension of the form (3.20) such that  $\dim(H^0(C, E)) = 1$ . It is enough to show that  $\dim(p_1(Y)) \leq \dim(H^1(C, L^*)) - 1$ .

Let  $Y' \subseteq Y$  be such that  $(a, \delta) \in Y' \Leftrightarrow \delta : \mathcal{O}_C \hookrightarrow L$  and  $\tilde{\delta}(a) = 0$ . Since  $C$  is a stable curve [10, Definition I.I], the dualizing sheaf  $\omega_C$  is locally free [10, Theorem 1.2]. So if

$$\delta : \mathcal{O}_C \hookrightarrow L,$$

is an injective map, then tensoring by  $\omega_C$ , we get an injective map

$$\omega_C \hookrightarrow L \otimes \omega_C.$$

So the induced map

$$H^1(C, L^*) \xrightarrow{\tilde{\delta}} H^1(C, \mathcal{O}_C)$$

will be surjective and

$$\dim(\ker(\tilde{\delta})) = \dim(H^1(C, L^*)) - g > 0. \quad (3.22)$$

It is clear from (3.22) that  $Y'$  is non-empty. We claim that  $Y'$  is open in  $Y$ .

Consider the second projection

$$p_2 : H^1(C, L^*) \times H^0(C, L) \rightarrow H^0(C, L).$$

As  $p_2^{-1}(V)$  is closed in  $H^1(C, L^*) \times H^0(C, L)$ ,  $p_2^{-1}(V) \cap Y$  is closed in  $Y$ , where  $V$  is as defined in (3.7).

Clearly  $p_2^{-1}(V) \cap Y = Y \setminus Y'$ . Therefore  $Y'$  is open in  $Y$ . We claim  $\dim(Y') = \dim(Y)$ .

If  $Y$  is irreducible, or if every irreducible component of  $Y$  intersects  $Y'$ , then we are done. Otherwise let  $Y_1 \subset Y$  be an irreducible component of  $Y$  such that  $Y_1 \cap Y' = \emptyset$ . Consider the map  $p_{2|Y_1} : Y_1 \rightarrow \overline{p_2(Y_1)}$ , where  $\overline{p_2(Y_1)}$  is the closure of  $p_2(Y_1)$  in  $H^0(C, L)$ . Since  $Y_1$  is irreducible,  $\overline{p_2(Y_1)}$  is also irreducible, and being a closed sub-variety of the affine space  $H^0(C, L)$ , it is an affine variety. Let  $\mathcal{U} \subset Y_1$  be an affine open and consider the map  $p_{2|_{\mathcal{U}}} : \mathcal{U} \rightarrow \overline{p_2(Y_1)}$ . This map is clearly a dominant map of irreducible affine varieties. Hence

there exists an open subset  $W_1$  of  $\overline{p_2(Y_1)}$  such that  $W_1 \subset p_2(Y_1)$ . Now further restricting  $p_2$  to  $p_2^{-1}(W_1)$  we get a surjective map  $p_2^{-1}(W_1) \rightarrow W_1$  of irreducible varieties. Therefore by [12, Theorem 1.25 (ii), Page-75], there exists an open subset  $W_2$  in  $W_1$  such that

$$\dim(p_2^{-1}(\phi)) = \dim(p_2^{-1}(W_1)) - \dim(W_1), \quad (3.23)$$

for all  $\phi \in W_2$ . But it is clear that  $\dim(p_2^{-1}(W_1)) = \dim(Y_1)$  and  $\dim(W_1) = \dim(\overline{p_2(Y_1)})$ . So the equation (3.23) becomes

$$\dim(p_2^{-1}(\phi)) = \dim(Y_1) - \dim(\overline{p_2(Y_1)}),$$

for all  $\phi \in W_2$ . So we have

$$\dim(Y_1) = \dim(p_2^{-1}(\phi)) + \dim(\overline{p_2(Y_1)}), \quad (3.24)$$

for all  $\phi \in W_2$ . (In equations (3.23) and (3.24), by  $p_2^{-1}(\phi)$  we mean  $p_2^{-1}(\phi) \cap Y$ ). We know that  $p_2^{-1}(\phi) \cap Y = \ker \tilde{\phi}$ . So by equation (3.24), to find  $\dim(Y_1)$  (or at least an "optimal" upper bound for  $\dim(Y_1)$ ), we have to find  $\dim(\overline{p_2(Y_1)})$  and  $\dim(\ker \tilde{\phi})$  for some  $\phi \in W_2$ .

Now since  $Y_1 \cap Y' = \emptyset$ ,  $p_2(Y_1) \subset V$ , where  $V$  is as defined in the equation (3.7). So there exists a  $\gamma$  such that  $\overline{p_2(Y_1)} \subset \overline{V_\gamma}$ . Let  $\gamma'$  be a proper subset of  $\{1, 2, \dots, N\}$  such that

$$\dim(\overline{V_{\gamma'}}) = \min \{\dim(\overline{V_\gamma}) \mid p_2(Y_1) \subset \overline{V_\gamma}\}.$$

Then  $p_2(Y_1)$  has to intersect  $V_{\gamma'}$  itself, for otherwise,  $p_2(Y_1)$  will be completely inside a smaller dimensional  $\overline{V_\gamma}$  which contradicts the fact that  $\overline{V_{\gamma'}}$  is minimum dimensional among all such  $\overline{V_\gamma}$ . Since  $V_{\gamma'}$  is open in its closure,  $V_{\gamma'} \cap \overline{p_2(Y_1)}$  is open in  $\overline{p_2(Y_1)}$ . So it has to intersect  $W_2$  and  $W_2 \cap V_{\gamma'} \cap \overline{p_2(Y_1)}$  is non-empty, open and is a subset of  $p_2(Y_1)$ . We denote this open set by  $W_3$ . Clearly  $\dim(\overline{p_2(Y_1)}) = \dim(W_3)$ . So the equation (3.24) becomes -

$$\dim(Y_1) = \dim(p_2^{-1}(\phi)) + \dim(W_3), \quad (3.25)$$

where  $\phi \in W_3$ .

Let  $\phi = (t_1, t_2, \dots, t_N) \in W_3$  be arbitrary. Since  $W_3 \subset V_{\gamma'}$ ,  $t_i = 0$  for  $i$  not in  $\gamma'$  and  $t_i \neq 0$  for  $i \in \gamma'$ . So for  $i \in \gamma'$ ,  $t_i : \mathcal{O}_{C_i} \rightarrow L_i$  is injective and the induced map  $\tilde{t}_i : H^1(C, L^*) \rightarrow H^1(C, \mathcal{O}_C)$  is surjective. Now consider the following commutative diagram -

$$\begin{array}{ccc} H^1(C, L^*) & \xrightarrow{\tilde{\phi}} & H^1(C, \mathcal{O}_C) \\ \downarrow & & \downarrow \\ \bigoplus_{j=1}^N H^1(C_j, L_j^*) & \xrightarrow{(\tilde{t}_1, \dots, \tilde{t}_N)} & \bigoplus_{j=1}^N H^1(C_j, \mathcal{O}_{C_j}). \end{array}$$

Both the vertical arrows in the above diagram are surjective because they are gotten by taking the long exact sequence corresponding to the short exact sequences -

$$0 \rightarrow L^* \rightarrow \bigoplus_{j=1}^N \alpha_{j*}(L_j^*) \rightarrow T' \rightarrow 0,$$

and

$$0 \rightarrow \mathcal{O}_C \rightarrow \bigoplus_{j=1}^N \alpha_{j*}(\mathcal{O}_{C_j}) \rightarrow \tilde{T} \rightarrow 0$$

respectively and observing that  $T'$  and  $\tilde{T}$  are supported only at the nodal points  $P_1, \dots, P_{N-1}$ , and hence  $H^1(C, T') = 0 = H^1(C, \tilde{T})$ . Since  $\tilde{t}_i$  is surjective for  $i \in \gamma'$ , we have

$\dim (\text{Im } (\tilde{t}_i)) = \dim (H^1(C_i, \mathcal{O}_{C_i})) = g_i$ , for such  $i$ , and  $\dim (\text{Im } (\tilde{t}_i)) = 0$ , for  $i$  not in  $\gamma'$ . Now since the above diagram commutes, we have  $\dim (\text{Im } (\tilde{\phi})) \geq \sum_{i \in \gamma'} g_i$ . This implies

$$\dim (\text{Ker } (\tilde{\phi})) \leq \dim (H^1(C, L^*)) - \sum_{i \in \gamma'} g_i, \quad (3.26)$$

for all  $\phi \in W_3$ . Also since  $W_3 \subset V_{\gamma'}$ , we have  $\dim (W_3) \leq \sum_{i \in \gamma'} g_i$  by Lemma 3.5. Using this and the inequality (3.26) in the equation (3.25), we get

$$\dim (Y_1) \leq \dim (H^1(C, L^*)).$$

So if  $Y_1$  is an irreducible component of  $Y$  such that  $Y' \cap Y_1 = \emptyset$ , then by the above arguments

$$\dim (Y_1) \leq \dim (H^1(C, L^*)). \quad (3.27)$$

Now by using similar arguments as above and the equation (3.22), one can prove

$$\begin{aligned} \dim (Y') &= \dim (H^0(C, L)) + \dim (H^1(C, L^*)) - g \\ &= \dim (H^1(C, L^*)). \end{aligned}$$

This implies by (3.27) that  $\dim (Y) = \dim (H^1(C, L^*))$ . This proves our claim that  $\dim (Y) = \dim (Y') = \dim (H^1(C, L^*))$ .

Now, if  $a \in p_1(Y)$ , then  $\dim (p_1^{-1}(a) \cap Y) \geq 1$  since  $(a, \delta) \in Y \Leftrightarrow (a, \lambda\delta) \in Y$  for every non-zero scalar  $\lambda$ . So

$$\begin{aligned} \dim (p_1(Y)) &\leq \dim (Y) - 1 \\ &= \dim (H^1(C, L^*)) - 1. \end{aligned}$$

This proves the first part of the lemma.

Now let

$$B = \{a \in H^1(C, L^*) \mid \dim (H^0(C, \tilde{\mathcal{E}}_a)) = 1\},$$

where  $\tilde{\mathcal{E}}$  is as in Lemma 3.7. Then by semi-continuity theorem,  $B$  is clearly a  $k^*$ -invariant open subset of  $H^1(C, L^*)$ . Let  $W$  be as in Remark 3.6 above. Then  $B \cap W \neq \emptyset$  as both  $B$  and  $W$  are Zariski-open subsets of an affine space. Since  $B$  and  $W$  are  $k^*$ -invariant,  $B \cap W$  is also  $k^*$ -invariant. Therefore  $B$  meets  $A$  also.  $\square$

**Remark 3.10.** (a) Let  $L$  be as above and  $F \subset L$  be a sub-sheaf. Now let  $t \geq 1$  be an integer and  $\gamma = \{i_1, i_2, \dots, i_t\} \subset \{1, 2, \dots, N\}$  be a proper subset such that  $i_1 < i_2 < \dots < i_t$ . Suppose  $F$  is such that  $\text{rk}(F_{i_1}), \text{rk}(F_{i_2}), \dots, \text{rk}(F_{i_t})$  are all equal to one and  $\text{rk}(F_i) = 0$  for  $i \neq i_1, i_2, \dots, i_t$ . Then it is clear that  $H^0(C, F) \subset \overline{V_\gamma}$ , where  $V_\gamma$  is as defined in equation (3.6). So we have

$$\begin{aligned} h^0(C, F) &\leq \dim (\overline{V_\gamma}) \\ &\leq \sum_{j=1}^t g_{i_j}. \end{aligned} \quad (3.28)$$

The last inequality follows from Lemma 3.5. Since  $\chi(F) \leq h^0(C, F)$ , from the inequality 3.28 we have

$$\chi(F) \leq \sum_{j=1}^t g_{i_j}. \quad (3.29)$$

(b) Let  $E$  be a locally free sheaf on  $C$  of rank two such that  $\dim (H^0(C, E)) = 1$ . Let  $G \subset E$  be a subsheaf such that its Euler characteristic is positive. Then  $\dim (H^0(C, G)) = 1$ . Moreover,

if  $E$  is an extension as in (3.20), then the map  $\mathcal{O}_C \rightarrow E$  factors through  $G$  because the sections of  $E$  are same as sections of  $G$ .

**Lemma 3.11.** *Let  $L$  be as above and  $E$  be an extension as in (3.20) such that  $\dim (H^0(C, E)) = 1$ . Then  $E$  is stable.*

*Proof.* Let  $G \subset E$  be a proper sub-sheaf. Since the weights are chosen in such a way that semi-stability coincides with stability, it is enough to prove

$$\chi(G) \leq \left( \sum_{j=1}^N w_j \operatorname{rk} (G_j) \right) \frac{\chi(E)}{2}. \quad (3.30)$$

By the choice of  $E$  and  $L$  it is clear that  $\chi(E) = 1$ . So we have to prove

$$\chi(G) \leq \frac{(\sum_{j=1}^N w_j \operatorname{rk} (G_j))}{2}.$$

We prove this by considering all possible cases for  $G$ .

**Case 1 :** Suppose  $G$  is such that  $\operatorname{rk} (G_i) = 2$  for all  $i$ . So we have to prove  $\chi(G) \leq 1$  in this case.

Suppose  $\chi(G) > 1$ . Then  $\dim (H^0(C, G)) > 1 = \dim (H^0(C, E))$ . But this is not possible as  $G \subset E$ . So we are done.

**Case 2 :** Suppose  $\operatorname{rk} (G_j) = 1$  for all  $j$ . In this case we have to prove  $\chi(G) \leq \frac{1}{2}$ .

Suppose  $\chi(G) > \frac{1}{2}$ , then by Remark 3.10,  $\dim (H^0(C, G)) = 1$  and the map  $\mathcal{O}_C \hookrightarrow E$  factors through  $G$ . Let us denote  $(\frac{G}{\mathcal{O}_C})$  by  $F$  for notational convenience. Since  $F$  is a subsheaf of  $(\frac{E}{\mathcal{O}_C}) = L$ , it implies  $F$  is either torsion free or zero. But the fact that  $\operatorname{rk} (G_j) = 1$  for all  $j$  forces  $F$  to be zero, for otherwise,  $F$  is supported at finitely many points and so it cannot be a subsheaf of  $L$ . This implies  $G \cong \mathcal{O}_C$  which gives a contradiction to our assumption that  $\chi(G) > \frac{1}{2}$ .

**Case 3 :** Suppose  $\operatorname{rk} (G_j) = 0$  for some  $j$ . We want to prove  $\chi(G) \leq \frac{(\sum_{j=1}^N w_j \operatorname{rk} (G_j))}{2}$ . Suppose  $\chi(G) > \frac{(\sum_{j=1}^N w_j \operatorname{rk} (G_j))}{2}$ . This implies  $\chi(G)$  is positive and so by the arguments in the Remark 3.10,  $\dim (H^0(C, G)) = 1$  and the map  $\mathcal{O}_C \hookrightarrow E$  factors through  $G$ . So we have  $\mathcal{O}_{C,p} \hookrightarrow G_p$  for each  $p \in C$ . In particular  $\mathcal{O}_{C,p} \hookrightarrow G_p$  for each smooth point  $p \in C_j$ . This contradicts the fact that  $\operatorname{rk} (G_j) = 0$  for some  $j$ .

**Case 4 :** Suppose  $i_1, i_2, \dots, i_t$  are indices in the increasing order such that  $\operatorname{rk} (G_{i_1}), \operatorname{rk} (G_{i_2}), \dots, \operatorname{rk} (G_{i_t})$  are all equal to two and  $\operatorname{rk} (G_j) = 1$  for  $j \neq i_1, i_2, \dots, i_t$  (here we are assuming that  $G_j$ 's are of mixed rank and  $\operatorname{rk} (G_j) \neq 0$  for all  $j$ ). Again, to prove the required result, if we assume on the contrary that  $\chi(G) > \frac{(\sum_{j=1}^N w_j \operatorname{rk} (G_j))}{2}$ , then the same arguments as before say that the map  $\mathcal{O}_C \hookrightarrow E$  factors through  $G$ . Denoting  $(\frac{G}{\mathcal{O}_C})$  by  $F$  as

before, and using the inequality 3.29 in the equation  $\chi(G) = \chi(\mathcal{O}_C) + \chi(F)$ , we get

$$\begin{aligned}
 \chi(G) &\leq \chi(\mathcal{O}_C) + \sum_{j=1}^t g_{i_j} \\
 &= 1 - g + \sum_{j=1}^t g_{i_j} \\
 &= 1 - \sum_{k \neq i_j} g_k.
 \end{aligned} \tag{3.31}$$

Since each  $g_k \geq 2$ , the inequality 3.31 implies that  $\chi(G)$  is negative. This is a contradiction. So we are done.  $\square$

**Remark 3.12.** (a) In this section we have assumed  $\chi = 1$  and  $\chi_i$ 's satisfy (3.1) and 3.2. So, for example, all the results of this section are valid if  $L$  is of type  $(1, 2, 2, \dots, 2)$  or  $(1, 3, 1, 3, \dots, 1, 3, 1)$  or  $(1, 3, 1, 2, 2, \dots, 2)$  and so on.

(b) Suppose  $L$  is of type  $(\chi_1, \chi_2, \dots, \chi_N)$  and

$$0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow L \rightarrow 0$$

is an exact sequence, then  $\chi_j$ 's are precisely the Euler characteristics of  $E_j$ 's. Since  $\chi(E) = \sum_{j=1}^N \chi_j - 2(N-1)$ , if  $\chi(E)$  is odd, then  $\sum_{j=1}^N \chi_j$  should be odd. This implies that the cardinality of the set

$$X = \{\chi_j \mid j \in \{1, 2, \dots, N\} \text{ and } \chi_j \text{ is an odd integer}\} \tag{3.32}$$

is an odd number.

In the proof of Lemma 3.9, we saw that the set  $B = \{a \in H^1(C, L^*) \mid \dim(H^0(C, \tilde{\mathcal{E}}_a)) = 1\}$  is a  $k^*$ -invariant nonempty open subset of  $H^1(C, L^*)$  and  $B \cap A$  is a non-empty open subset of the affine space  $A$ . Let  $S = B \cap A$ . Now, if  $a \in B$ , then by Lemma 3.11, it is clear that  $\tilde{\mathcal{E}}_a$  is stable. So  $S$  is a non-empty open subset of the affine space  $A$  consisting of stable rank two locally free sheaves  $\tilde{\mathcal{E}}_a$  such that  $\dim(H^0(C, \tilde{\mathcal{E}}_a)) = 1$ . Since  $S \subset A$  is Zariski-open,  $\dim(S) = \dim(A) = 3g - 3$ .

**Lemma 3.13.** Let  $L$  be an invertible sheaf on  $C$  of any type mentioned above. Then there exists a non-empty open subset  $S$  of an affine space and a locally free sheaf  $\mathcal{E}'$  of rank two on  $S \times C$  such that

- (i)  $\dim(S) = 3g - 3$ ,
- (ii)  $\mathcal{E}'_s \cong \mathcal{E}'_t \Leftrightarrow s = t$ ,
- (iii) for all  $s \in S$ ,  $\mathcal{E}'_s$  is stable and  $\Lambda^2(\mathcal{E}'_s) = L$ .

*Proof.* Consider the sheaf  $\tilde{\mathcal{E}}$  on  $H^1(C, L^*) \times C$  obtained in (3.19) and restrict it to  $S \times C$ , where  $S$  is as defined just above. Let  $\mathcal{E}' = \tilde{\mathcal{E}}|_{S \times C}$ . We have already seen that  $\dim(S) = 3g - 3$ . By the definition of  $S$  (more precisely, by the definition of  $A$ ) and Remark 3.8, it is clear that  $\mathcal{E}'_s \cong \mathcal{E}'_t \Leftrightarrow s = t$ . Again by the definition of  $S$  it is clear that for all  $s \in S$ ,  $\mathcal{E}'_s$  is stable and  $\Lambda^2(\mathcal{E}'_s) = L$ . This proves the Lemma.  $\square$

## 4. RATIONALITY

Suppose  $\chi = 1$ ,  $C$ ,  $\underline{w}$ ,  $L$  and  $\chi_j$ 's are as before. Let  $M_{(\chi_1, \dots, \chi_N)}(L)$  denote the collection of all vector bundles in  $M(2, \underline{w}, \chi = 1)$  with determinant  $L$ . Since  $M(2, \underline{w}, \chi = 1)$  is a coarse moduli space, by Lemma 3.13, we have an injective morphism

$$f: S \rightarrow M(2, \underline{w}, \chi = 1), \quad (4.1)$$

where  $S$  is as in the Lemma 3.13. But the image of  $f$  lands in  $M_{(\chi_1, \dots, \chi_N)}(L)$ . Since  $S$  and  $M_{(\chi_1, \dots, \chi_N)}(L)$  are both smooth varieties of same dimension and we are in characteristic zero, it implies that  $f$  is birational. So  $M_{(\chi_1, \dots, \chi_N)}(L)$  is rational.

**Remark 4.1.** Let  $U' \subset M_{(\chi_1, \dots, \chi_N)}(L)$  be the open subset consisting of all vector bundles  $E$  such that  $E_i$  is semi-stable for each  $i$ . By [11, Step 2],  $U' \neq \emptyset$ . Let  $U = f^{-1}(U')$ . Then  $U \neq \emptyset$  as  $f$  is birational. Since  $U \subset S$  is open,  $\dim(U) = 3g - 3$ .

Let  $\mathcal{E}'$  be as in Lemma 3.13. Then  $\mathcal{E}'|_U$  is a locally free sheaf of rank two on  $U \times C$  such that for each  $u \in U$ ,  $\mathcal{E}'_u$  is stable and for each  $i$ ,  $\mathcal{E}'_u|_{C_i}$  is semi-stable. Clearly the restriction map

$$f: U \rightarrow M_{(\chi_1, \dots, \chi_N)}(L) \quad (4.2)$$

is birational.

This remark is important because, stability of an arbitrary vector bundle  $E$  on  $C$  does not guarantee the semi-stability of  $E_i$  on  $C_i$ . But there is a non-empty open set in the moduli space consisting of stable vector bundles whose restriction to each component is semi-stable.

With these in mind, we now state and prove the main proposition-

**Proposition 4.2.** Let  $\chi$  be an odd integer and  $C, \underline{w}$  be as mentioned above. Let  $\chi_1, \dots, \chi_{N-1}$  be the integers satisfying the inequalities -

$$\left(\sum_{j=1}^i w_j\right)\chi - \sum_{j=1}^{i-1} \chi_j + 2(i-1) < \chi_i < \left(\sum_{j=1}^i w_j\right)\chi - \sum_{j=1}^{i-1} \chi_j + 2i,$$

and  $\chi_N$  be the integer such that

$$\chi = \sum_{j=1}^N \chi_j - 2(N-1).$$

Let  $L = (L_1, \dots, L_N)$  be an invertible sheaf on  $C$  of type  $(\chi_1, \chi_2, \dots, \chi_N)$ . Then there exists a non-empty open subset  $U$  of an affine space and a locally free sheaf  $\mathcal{E}$  on  $U \times C$  of rank two such that

- (i)  $\dim(U) = 3g - 3$ ,
- (ii)  $\mathcal{E}_u \cong \mathcal{E}_t \Leftrightarrow u = t$ ,
- (iii) for all  $u \in U$ ,  $\mathcal{E}_u$  is stable and  $\Lambda^2(\mathcal{E}_u) = L$ .

*Proof.* Let  $i_1, i_2, \dots, i_t$  be the indices in the increasing order such that  $\chi_{i_1}, \chi_{i_2}, \dots, \chi_{i_t}$  are odd integers and  $\chi_j$ 's are even integers if  $j \neq i_1, \dots, i_t$ . So  $t$  is an odd number (see Remark 3.12(b) and equation (3.32)).

If  $\chi = 1$  and each  $L_j$  is globally generated, then by Lemma 3.13 and the above arguments, we are done. Also when  $\chi = 1$ , among all the  $2^{N-1}$  choices for the tuple  $(\chi_1, \dots, \chi_N)$ , there exists a choice for which  $\chi_{i_j} = 1$  if  $j$  is odd,  $\chi_{i_j} = 3$  if  $j$  is even and  $\chi_i = 2$  if  $i$  does not

belong to  $\{i_1, i_2, \dots, i_t\}$ . Now since  $L = (L_1, \dots, L_N)$  is of type  $(\chi_1, \dots, \chi_N)$ , and  $\chi_{i_j}$ 's are odd for  $j = 1, \dots, t$ ,  $\deg(L_{i_j}) = \chi_{i_j} - 2(1 - g_{i_j})$  will be odd for each  $i_j$  and  $\deg(L_r) = \chi_r - 2(1 - g_r)$  will be even for  $r \neq i_j$ . For notational convenience, we write  $\deg(L_{i_j}) = 2l_{i_j} - 1$  for  $j = 1, \dots, t$ , and  $\deg(L_r) = 2l_r$  for  $r \neq i_j$ , where  $l_{i_j} = \frac{\chi_{i_j} - 2(1 - g_{i_j}) + 1}{2}$  for each  $i_j$ , and  $l_r = \frac{\chi_r - 2(1 - g_r)}{2}$  for each  $r \neq i_j$ . Let  $M$  be an invertible sheaf on  $C$  such that

$$\deg(M_r) = (l_r - g_r) \text{ for } r \neq i_2, i_4, \dots, i_{t-1},$$

$$\deg(M_r) = (l_r - g_r - 1) \text{ for } r \in \{i_2, i_4, \dots, i_{t-1}\}, \text{ and}$$

$L_r \otimes M_r^{-2}$  is globally generated for  $r = 1, 2, \dots, N$  (see [1, Remark 4.2] for the existence of such an  $M$ ).

It is clear that

$$\deg(L_r \otimes M_r^{-2}) = 2g_r \text{ for } r \neq i_1, i_2, \dots, i_t,$$

$$\deg(L_r \otimes M_r^{-2}) = 2g_r - 1 \text{ for } r \in \{i_1, i_3, \dots, i_t\}, \text{ and}$$

$$\deg(L_r \otimes M_r^{-2}) = 2g_r + 1 \text{ for } r \in \{i_2, i_4, \dots, i_{t-1}\}.$$

So by Lemma 3.13 and Remark 4.1, there exists a non-empty open subset  $U$  of an affine space and a locally free sheaf  $\mathcal{E}'$  on  $U \times C$  such that

$$(i) \dim(U) = 3g - 3,$$

$$(ii) \mathcal{E}'_u \cong \mathcal{E}'_t \Leftrightarrow u = t,$$

$$(iii) \text{ for all } u \in U, \mathcal{E}'_u \text{ is stable and } \Lambda^2(\mathcal{E}'_u) = L \otimes M^{-2}.$$

Let  $\mathcal{E} = \mathcal{E}' \otimes p_C^*(M)$ . Then  $\mathcal{E}_u = \mathcal{E}'_u \otimes M$  and  $\Lambda^2(\mathcal{E}_u) = \Lambda^2(\mathcal{E}'_u) \otimes M^2 = L$  for all  $u \in U$ .

Now since  $\mathcal{E}'_u$  is stable, by the Remark 4.1,  $\mathcal{E}'_u|_{C_i}$  is semi-stable for all  $i$ . This means  $\mathcal{E}'_u|_{C_i} \otimes M_i$  is semi-stable for all  $i$  and  $\mathcal{E}'_u|_{C_j} \otimes M_j$  is stable for  $j = i_1, i_2, \dots, i_t$ . This proves that  $\mathcal{E}_u$  is stable (see [11, Step 2]).

This proves the proposition.  $\square$

From the Proposition, we can conclude that the sheaf  $\mathcal{E}$  on  $U \times C$  induces a morphism

$$f : U \rightarrow M_{(\chi_1, \dots, \chi_N)}(L).$$

By (ii) of the Proposition,  $f$  is injective. Since  $U$  and  $M_{(\chi_1, \dots, \chi_N)}(L)$  are smooth varieties of same dimension and we are in characteristic zero, it implies that  $f$  is birational. So  $M_{(\chi_1, \dots, \chi_N)}(L)$  is rational.

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